# Restricted Permutations

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**Abstract**

Restricted permutations are those constrained by having to avoid subsequences ordered in various prescribed ways. They have functioned as a convenient descriptor for several sets of permutations which arise naturally in combinatorics and computer science. We study the partial order on permutations (and more general sequences) that underlies the idea of restriction and which gives rise to sets of sequences closed under taking subsequences. In applications, the question of whether a closed set has a finite basis is often considered. We give a family of sets that have a finite basis, and then use them to study the inverse problem of describing a closed set from its basis. We give enumeration results in all cases where the basis consists of a permutationof length 3 and a permutation of length 4. The paper is an extended version of [2] in that it contains full proofs of the results presented there which have also been obtained by West [15].

## 1 General setting

The study of permutations which are constrained by not having one or more subsequences ordered in various prescribed ways has been motivated both by its combinatorial difficulty and by its appearance in some data structuring problems in Computer Science. The fundamental relation that underpins this study is *involvement* which captures the idea of one sequence being ordered in the same way as a subsequence of another. Two numerical sequences $\pi = [p_1, p_2, \ldots, p_m]$ and $\rho = [r_1, r_2, \ldots, r_m]$ of the same length are said to be *order isomorphic* if, for all $i, j$, $p_i < p_j$ if and only if $r_i < r_j$. Order isomorphism is clearly an equivalence relation on sequences. Throughout this paper we shall consider only sequences of distinct elements. Every such sequence of length $n$ is order isomorphic to a unique permutation of $1, 2, \ldots, n$ and, for this reason, most of our results are stated for permutations. Unless otherwise stated "permutation" will always mean an arrangement of $1, 2, \ldots, n$ for some $n$. Generally, sequences will be denoted by Greek letters and their elements by Roman letters.

If $\pi$ and $\sigma$ are sequences then $\pi$ is said to be *involved* in $\sigma$ if $\pi$ is order isomorphic to a subsequence $\rho$ of $\sigma$; we write $\pi \preceq \sigma$. For example, $[2, 3, 1, 4] \preceq$

$[6,3,5,7,2,4,1,8]$ because of the subsequence $[3,5,2,8]$ in the second permutation. For permutations on a small number of symbols it is often convenient to omit the brackets and commas and write $2314 \preceq 63572418$.

A map $\alpha$ from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$ is said to be *monotonic* if $\alpha(i) < \alpha(j)$ whenever $i < j$. Monotonic maps allow us to describe the terms 'subsequence' and 'order isomorphism' using functional composition (which we write from left to right). Suppose that $\pi$ and $\sigma$ are permutations. A sequence of positive integers is order isomorphic to $\pi$ if and only if it has the form $\pi\alpha$ where $\alpha$ is a monotonic map. Furthermore a sequence is a subsequence of $\sigma$ if and only if it has the form $\beta\sigma$ with $\beta$ a monotonic map. In particular $\pi \preceq \sigma$ if and only if there exist monotonic maps $\alpha, \beta$ such that $\pi\alpha = \beta\sigma$

A set $\mathcal{X}$ of permutations is said to be *closed* if, whenever $\sigma \in \mathcal{X}$ and $\pi \preceq \sigma$, then $\pi \in \mathcal{X}$. The archetypal example of a closed set is the set of *stack sortable* permutations. A sequence is stack sortable if, when it is presented as input to a stack and subjected to an appropriate series of 'push' and 'pop' operations, the stack can produce the elements in ascending order. It is evident that if a sequence is stack sortable then so is any sequence order isomorphic to it and also any subsequence. In particular, if $\sigma$ is a stack sortable permutation and $\pi \preceq \sigma$ then $\pi$ is also stack sortable.

Stack sortable permutations were first studied in [6] where two results were proved which have continued to inspire the study of closed sets. The first is that a permutation is stack sortable if and only if it does not involve the permutation 231. The second is that the number of stack sortable permutations of length $n$ is $\binom{2n}{n}/(n+1)$. The first of these results motivates the definition of the 'basis' of a closed set below and allows several combinatorial results in the literature to be described uniformly. We shall survey some of these below and give some new results in the next section. The second result has been generalised to a number of other closed sets and we shall present some further results in section 3. At this point however it is convenient to introduce the terminology $\mathcal{X}_n$ to denote the subset of $\mathcal{X}$ whose permutations have length $n$.

If $\mathcal{X}$ is closed let $\mathcal{X}^\star$ denote the set of permutations, minimal with respect to $\preceq$, that do not belong to $\mathcal{X}$. In turn, $\mathcal{X}^\star$ determines $\mathcal{X}$ as $\{\alpha | \beta \not\preceq \alpha \text{ for all } \beta \in \mathcal{X}^\star\}$. The set $\mathcal{X}^\star$ is called the *basis* of $\mathcal{X}$. In this terminology the set of stack sortable permutations has the basis $\{231\}$.

Many natural closed sets of permutations $\mathcal{X}^\star$ have a very simple basis. For example:

- If $\mathcal{X}$ is the set of permutations that can be sorted by a restricted input deque then $\mathcal{X}^\star = \{4231, 3241\}$ [6, 7, 11].

- If $\mathcal{X}$ is the set of permutations that can be expressed as the interleaving of two increasing subsequences then $\mathcal{X}^\star = \{321\}$ [6].

- If $\mathcal{X}$ is the set of permutations that can be expressed as the interleaving of an increasing subsequence and a decreasing subsequence then $\mathcal{X}^\star = \{3412, 2143\}$ (see [5, 11]).

- If $\mathcal{X}$ is the set of permutations that can be obtained by a 'riffle' shuffle of a deck of cards $1, 2, \ldots, n$ then $\mathcal{X}^\star = \{321, 2143, 2413\}$ (see the proof of Proposition 3.6 below).
- If $\mathcal{X}$ is the set of all 'separable' permutations [3] then $\mathcal{X}^\star = \{3142, 2413\}$ (see also [9] where these permutations are considered in the context of 'bootstrap percolation').

However, there are also many closed sets whose basis is not simple to describe nor even finite; examples of closed sets with an infinite basis are given in [13, 7]. The converse problem of describing the closed set defined by a given basis $\mathcal{B}$ has also attracted some study; we call this closed set $\mathcal{A}(\mathcal{B})$, the letter $\mathcal{A}$ recalling that $\mathcal{A}(\mathcal{B})$ is the set of permutations which *avoid* every permutation of $\mathcal{B}$. It is trivial to find $\mathcal{A}(\mathcal{B})$ if the basis elements are permutations of lengths 1 or 2 ($\mathcal{A}(\{1\})$ is empty, $\mathcal{A}(\{21\})$ consists only of identity permutations, etc.). In [10] Simion and Schmidt gave complete descriptions of closed sets whose bases consist of sets of permutations of length 3. West [14] and Stankova [11, 12] have begun the study of bases comprising permutations of length 4 but this is still very incomplete.

Another theme running through the above works is enumeration:– finding the number of permutations of each length in a closed set. We let $\mathcal{A}_n(\mathcal{B})$ be the set of permutations in $\mathcal{A}(\mathcal{B})$ of length $n$. Occasionally it is necessary to consider the permutations of length $n$ of some set other than $\{1, 2, \ldots, n\}$ which avoid $\mathcal{B}$ but this set has the same size as $\mathcal{A}_n(\mathcal{B})$.

In all this work it is very useful to take advantage of some natural symmetries based on the following facts (which were first made explicit in [10]). If $\sigma$ is any permutation on $\{1, 2, \ldots, n\}$, let $\bar{\sigma}$ and $\sigma^\star$, respectively, denote the permutations obtained from $\sigma$ by replacing every element $s_i$ by $n + 1 - s_i$ and reversing the elements of $\sigma$. Also, as usual, let $\sigma^{-1}$ denote the permutation inverse of $\sigma$. Then

1. If $\pi \preceq \sigma$ then $\bar{\pi} \preceq \bar{\sigma}$
2. If $\pi \preceq \sigma$ then $\pi^\star \preceq \sigma^\star$
3. If $\pi \preceq \sigma$ then $\pi^{-1} \preceq \sigma^{-1}$

These 3 symmetries generate the dihedral group $D$ of order 8. It acts in a natural way on sets of permutations. As a direct consequence of the definitions we have

**LEMMA 1.1** *If $\lambda$ is any element of the symmetry group $D$ and $\mathcal{X}$ is any closed set of permutations with basis $\mathcal{X}^\star$ then $\lambda(\mathcal{X})$ is closed and has basis $\lambda(\mathcal{X}^\star)$. Furthermore, $|\mathcal{X}_n| = |\lambda(\mathcal{X})_n|$ for all $n$.*

As an example of the power of this lemma consider the problem of finding $\mathcal{A}(\sigma)$ when $\sigma$ has length 4. Although there are 24 such problems they fall into 7 symmetry classes under the action of $D$. According to the lemma $|\mathcal{A}_n(\sigma)| = |\mathcal{A}_n(\tau)|$ whenever $\sigma$ and $\tau$ are equivalent under $D$. Mysteriously, this equation sometimes holds when $\sigma$ and $\tau$ are not equivalent. Some reasons for this are given in [11, 12] but much remains to explain. As we shall see in section 3 there are numerous other equalities of this sort.

In section 2 of the paper we give some constructions and results for combining closed sets. We follow this with a discussion of a large family of closed sets each of which has a finite basis. These closed sets will feature in section 3 but we end section 2 by using them to solve a problem on riffle shuffles. Section 3 is devoted to the enumeration problem for closed sets defined by a given basis; we give a complete treatment of the case where the basis consists of a permutation of length 3 and a permutation of length 4.

# 2 Some finitely based sets

## 2.1 Combining closed sets

There are several ways in which two or more closed sets can be combined to give another closed set. This subsection reviews those combinations which are used later in the paper.

**THEOREM 2.1** *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are closed sets. Then $\mathcal{X}\cap\mathcal{Y}$ and $\mathcal{X}\cup\mathcal{Y}$ are also closed. Moreover if $\mathcal{X}$ and $\mathcal{Y}$ each have a finite basis then both $\mathcal{X}\cap\mathcal{Y}$ and $\mathcal{X}\cup\mathcal{Y}$ have a finite basis.*

PROOF That $\mathcal{X}\cap\mathcal{Y}$ and $\mathcal{X}\cup\mathcal{Y}$ are closed follows directly from the definitions. Now suppose that $\mathcal{X}=\mathcal{A}(S)$ and $\mathcal{Y}=\mathcal{A}(T)$ for finite sets $S$ and $T$. Since, obviously, $\mathcal{X}\cap\mathcal{Y}=\mathcal{A}(S\cup T)$ it follows that $\mathcal{X}\cap\mathcal{Y}$ has a finite basis.

Finally consider a permutation $\alpha$ in the basis of $\mathcal{X}\cup\mathcal{Y}$. Such a permutation belongs neither to $\mathcal{X}$ nor to $\mathcal{Y}$ and so has subsequences $\sigma$ and $\tau$ which are order isomorphic to permutations in $S$ and $T$ respectively. However, $\alpha$ is minimal and so no proper subsequence also has this property. Thus $\alpha$ must be the union of $\sigma$ and $\tau$ and so has bounded length. Therefore there are only finitely many possibilities for $\alpha$. ■

**THEOREM 2.2** *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are closed sets each with a finite basis. Let $[\mathcal{X},\mathcal{Y}]$ be the set of all permutations which are concatenations $\sigma\tau$ where $\sigma$ is order isomorphic to a permutation in $\mathcal{X}$ and $\tau$ is order isomorphic to a permutation in $\mathcal{Y}$. Then $[\mathcal{X},\mathcal{Y}]$ is closed. Moreover if $\mathcal{X}$ and $\mathcal{Y}$ are each finitely based then so is $[\mathcal{X},\mathcal{Y}]$*

PROOF It is evident that $[\mathcal{X},\mathcal{Y}]$ is closed. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are each finitely based and that $\alpha$ is a permutation in the basis of $[\mathcal{X},\mathcal{Y}]$. Let $\alpha=\sigma\tau k$ where $k$ is the last symbol of $\alpha$ and where (since $\alpha$ is minimal with respect to not belonging to $[\mathcal{X},\mathcal{Y}]$) we may presume that $\sigma$ and $\tau$ are order isomorphic to permutations of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Among all such decompositions for $\alpha$ choose the one with $\sigma$ of maximal length. Then, if $t$ is the first symbol of $\tau$, $\sigma t$

is not order isomorphic to a permutation in $\mathcal{X}$ and $\tau k$ is not order isomorphic to a permutation in $\mathcal{Y}$.

It follows that $\sigma t$ has a subsequence $\sigma' t$ order isomorphic to a permutation in the basis $S$ of $\mathcal{X}$ and $\tau k$ has a subsequence $\tau' k$ order isomorphic to a permutation in the basis $T$ of $\mathcal{Y}$. But then the subsequence $\sigma' t \tau' k$ (or $\sigma' \tau' k$ if $t$ is a symbol of $\tau'$) of $\alpha$ cannot be order isomorphic to a permutation of $[\mathcal{X}, \mathcal{Y}]$ and, by minimality of $\alpha$, must be $\alpha$ itself. Since $\sigma' t$ and $\tau' k$ are bounded in length (since $S$ and $T$ are finite), the length of $\alpha$ is also bounded. ■

## 2.2 Profile classes

If $A$ and $B$ are sets or sequences we write $A < B$ to denote that $a < b$ for all $a \in A, b \in B$. As a first use of this notation we define the *profile* of a permutation. If $\rho$ and $\pi$ are permutations then $\rho$ is said to have profile $\pi = [p_1 \dots p_m]$ if $\rho$ has a partition into segments $\rho = \rho_1 \dots \rho_m$ where $m$ is minimal subject to

1. each $\rho_i$ is a non-empty sequence of increasing consecutive symbols
2. $\rho_i < \rho_j$ if and only if $p_i < p_j$

For example, 34597812 has profile 2431 because of its segments $345, 9, 78, 12$. Clearly, a permutation determines its profile uniquely. Not every permutation can be a profile however; to be a profile the permutation must not contain any segment $t, t+1$.

**LEMMA 2.3** *If $\pi$ is a valid profile and has length $m$ then the number of permutations of length $n$ which have profile $\pi$ is $\binom{n-1}{m-1}$.*

PROOF If $\rho$ is a permutation with profile $\pi$ (by way of a decomposition $\rho = \rho_1 \dots \rho_m$) then $\rho$ is determined by the lengths of the $\rho_i$, i.e. by an ordered set of $m$ positive integers whose sum is $n$. Since every such composition of $n$ can arise in this way and there are $\binom{n-1}{m-1}$ such compositions the result follows. ■

We define a set $\Sigma$ of permutations to be *profile-closed* if all its members are valid profiles and, whenever $\beta$ is a valid profile with $\beta \preceq \alpha \in \Sigma$, then $\beta \in \Sigma$. The *profile closure* of a set of profiles is defined to be the smallest profile-closed set containing it. As an example, the profile closure of $\{2431\}$ is the profile-closed set $\{2431, 132, 321, 21, 1\}$.

**THEOREM 2.4** *If $\Sigma$ is a profile-closed set of permutations then $P(\Sigma)$, the set of permutations whose profile lies in $\Sigma$, is closed. Furthermore, if $\Sigma$ is finite then $P(\Sigma)$ has a finite basis.*

PROOF It follows from the definitions that, if $\rho$ has profile $\pi$ and $\lambda \preceq \rho$, then $\lambda$ has profile $\mu$ where $\mu \preceq \pi$. This proves the first part. For the second part let $\beta$ be a permutation on $1, 2, \dots, m$ in the basis of $P(\Sigma)$. Suppose that $\beta$ has two adjacent consecutive symbols $t, t+1$; then $\beta$ and $\beta - t$ have the same profile. However, $\beta - t$ is order isomorphic to a permutation in $P(\Sigma)$ and so its profile lies in $\Sigma$. Thus $\beta \in P(\Sigma)$ which is impossible. Hence no two adjacent symbols of $\beta$ can be consecutive.

The permutation $\beta - m$ can have at most two adjacent consecutive symbols (which, in $\beta$, were separated by $m$) and so $\beta - m$ has length at most 1 more than the length of its profile. But $\beta - m \in P(\Sigma)$ and so its profile lies in $\Sigma$. Therefore the length of $\beta$ is bounded and the proof is complete. ■

We shall appeal to these results in the next section. They may be generalised in several ways. We can, of course, consider profiles based on decreasing segments rather than increasing segments. More interestingly we can consider profiles where segments are allowed to be both increasing and decreasing; a similar finite basis result can be proved. We can also consider permutations with a profile where one or more of the increasing segments is of bounded length. In particular, in the next section we require, at one point, profiles where one of the segments has length 0 or 1; we shall show this by a superscript $^1$; so, for example, permutations with the (generalised) profile $13^12$ would be structured as $[1, 2, \ldots, k, n, k+1, \ldots, n-1]$ for some $k$.

## 2.3 Riffle shuffles

We have already mentioned, in section 1, the closed set of permutations obtained by a standard riffle shuffle of a deck of $n$ cards. These riffle shuffle permutations are, of course, just merges of cards $1, 2, \ldots, m$ (for some $m$) and cards $m+1, \ldots, n$. More generally we wish to consider $S_r$ the set of $r$-shuffles which are defined by cutting a deck into $r$ sections and interleaving these sections in any way. The inverse of an $r$-shuffle $\pi$ is, by definition, an ordering of the deck of cards from which the $r$-shuffle $\pi$ could restore the deck to its original order.

**LEMMA 2.5** *A permutation $\pi$ of length $n$ is an $r$-shuffle if and only if there exist partitions $\bigcup_{k=1}^{r} A_k$ and $\bigcup_{k=1}^{r} I_k$ of $\{1, \ldots, n\}$ such that*

1. *$I_k < I_{k+1}$ for all $k$*
2. *$\pi(A_k) = I_k$ for all $k$*
3. *$\pi|_{A_k}$ is monotonic increasing for all $k$*

PROOF An $r$-shuffle begins by dividing $\{1, \ldots, n\}$ into segments $I_1, \ldots, I_r$ satisfying property 1. When the segments are interleaved each set $I_k$ is distributed, without disturbing its order, into a set of positions $A_k$ of the resulting permutation $\pi$ and therefore conditions 2 and 3 hold. The converse is clear. ■

An immediate consequence of this lemma is a corresponding characterisation of the inverses of shuffles.

**LEMMA 2.6** *A permutation $\pi$ of length $n$ is the inverse of a $t$-shuffle if and only if there exist partitions $\bigcup_{k=1}^{t} B_k$ and $\bigcup_{k=1}^{t} J_k$ of $\{1, \ldots, n\}$ such that*

1. $J_k < J_{k+1}$ *for all* $k$
2. $\pi(J_k) = B_k$ *for all* $k$
3. $\pi|_{J_k}$ *is monotonic increasing for all* $k$

Notice that $\pi$ is the inverse of a $t$-shuffle if and only if $\pi$ has at most $t-1$ *descents* (positions $i$ where $\pi_i > \pi_{i+1}$). The number $S_t(n)$ of permutations of this type is the classical Simon Newcomb's problem (see p.213ff of [8]). Also notice that $S_t^{-1} = [\mathcal{I}, \mathcal{I}, \ldots]$ where $\mathcal{I}$ is the set of all identity permutations and so $S_t^{-1}$ and $S_t$ are finitely based by Theorem 2.2 and Lemma 1.1.

The main result of this subsection is a structure theorem for the closed set $S_r \cap S_t^{-1}$.

**THEOREM 2.7** *Let $\Sigma$ be the profile closure of the single permutation*

$$[1, r+1, 2r+1, \ldots, (t-1)r+1, 2, r+2, \ldots, (t-1)r+2, 3, \ldots, n]$$

*Then $P(\Sigma) = S_r \cap S_t^{-1}$.*

PROOF Suppose that $\pi \in S_r \cap S_t^{-1}$. Let $\{A_i\}_{k=1}^{r}, \{I_k\}_{k=1}^{r}, \{B_k\}_{k=1}^{t}, \{J_k\}_{k=1}^{t}$ be the sets defined and guaranteed by the previous two lemmas. Let $C_{hk} = A_h \cap J_k$ and $D_{hk} = I_h \cap B_k$.

Since $C_{hk}, C_{h+1,k} \subseteq J_k$, $\pi|_{J_k}$ is monotonic increasing, and

$$\pi(C_{hk}) = D_{hk} = I_h \cap B_k < I_{h+1} \cap B_k = D_{h+1,k} = \pi(C_{h+1,k})$$

we have $C_{hk} < C_{h+1,k}$. Furthermore $C_{rk} = A_r \cap J_k < A_1 \cap J_{k+1} = C_{1,k+1}$. Therefore

$$C_{11} < C_{21} < \ldots < C_{r1} < C_{12} < C_{22} < \ldots$$

Also, by a similar argument,

$$D_{11} < D_{12} < \ldots < D_{1s} < D_{21} < D_{22} < \ldots$$

It follows that the profile of $\pi$ is in the set $\Sigma$. This proves one half of the Theorem. The converse can be proved by reversing the foregoing argument. ■

In principle, this theorem allows the enumeration problem to be solved for any fixed $S_r \cap S_t^{-1}$. We illustrate this for the standard riffle shuffles ($2-$shuffles) in the next lemma.

**LEMMA 2.8** *The number of riffle shuffles of a deck of $n$ cards which can be restored by a riffle shuffle is $\binom{n+1}{3} + 1$.*

Proof According to Theorem 2.7 $S_2 \cap S_2^{-1} = P(\Sigma)$ where $\Sigma = \{1324, 213, 132, 21, 1\}$ is the profile closure of 1324. Therefore, by Lemma 2.3,

$$\begin{aligned} |(S_2 \cap S_2^{-1})_n| &= \binom{n-1}{3} + 2\binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0} \\ &= \binom{n+1}{3} + 1 \end{aligned}$$

■

# 3 Closed sets with a basis of two permutations of lengths 3 and 4

In this section we consider all closed sets which have a basis of two permutations, $\alpha$ of length 3, $\beta$ of length 4. Of the $144 = 3! \times 4!$ pairs of such permutations we may immediately reduce to a complete set of pairs inequivalent under the symmetry group $D$. There are 30 such pairs but 12 of them are degenerate in the sense that $\alpha \preceq \beta$ and therefore $\{\alpha, \beta\}$ is not a basis of a closed class. For the remaining 18 pairs the following table gives the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ or a recurrence relation they satisfy. Every pair $\alpha, \beta$ with $\alpha \not\preceq \beta$ is equivalent to one of these pairs.

| | $\alpha, \beta$ | $a_n = \lvert\mathcal{A}_n(\alpha, \beta)\rvert$ |
|---|---|---|
| 1 | $123, 4321$ | $0$ for $n \geq 7$ |
| 2 | $321, 2134$ | $n + \binom{n}{3} + \binom{n+1}{4}$ |
| 3 | $321, 1324$ | $1 + \binom{n}{2} + \binom{n+1}{5}$ |
| 4 | $132, 4321$ | $1 + \binom{n+1}{3} + 2\binom{n}{4}$ |
| 5 | $123, 4213$ | $3 \times 2^{n-1} - \binom{n+1}{2} - 1$ |
| 6 | $123, 3412$ | $2^{n+1} - 2n - 1 - \binom{n+1}{3}$ |
| 7 | $132, 4312$ | $(n-1)2^{n-2} + 1$ |
| 8 | $132, 4231$ | $(n-1)2^{n-2} + 1$ |
| 9 | $132, 3214$ | $a_n = 4a_{n-1} - 5a_{n-2} + 3a_{n-3}$ |
| 10 | $123, 3214$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 11 | $132, 1234$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 12 | $132, 4213$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 13 | $132, 4123$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 14 | $132, 3124$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 15 | $123, 2143$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 16 | $123, 3142$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 17 | $132, 2134$ | $a_n = 3a_{n-1} - a_{n-2}$ |
| 18 | $132, 3412$ | $a_n = 3a_{n-1} - a_{n-2}$ |

In the remainder of this section we give the main ideas behind this table in a series of propositions 3.1 to 3.18, one for each line of the table. Note that cases 10–18 all define the same sequence $1, 2, 5, 13, 34, \ldots$ of alternate Fibonacci numbers.

**PROPOSITION 3.1** *If $\alpha = 123$ and $\beta = 4321$ the values of $|\mathcal{A}_n(\alpha, \beta)|$ are $1, 2, 5, 13, 25, 25$ for $1 \leq n \leq 6$ and zero for $n \geq 7$.*

PROOF For $1 \leq n \leq 6$ this follows from direct enumeration whilst for $n \geq 7$ it is a special case of a theorem of Erdös and Szekeres [4]. ∎

The proofs of the next three propositions are all similar so we give details for the last only.

**PROPOSITION 3.2** *If $\alpha = 321$ and $\beta = 2134$ then $\mathcal{A}(\alpha, \beta)$ is the set of permutations whose profiles are in the profile closure of $14627^1 35^1$. Moreover, $|\mathcal{A}_n(\alpha, \beta)| = n + \binom{n}{3} + \binom{n+1}{4}$*

**PROPOSITION 3.3** *If $\alpha = 321$ and $\beta = 1324$ then $\mathcal{A}(\alpha, \beta)$ is the set of permutations whose profiles are in the profile closure of 21354 and 351624. Moreover, $|\mathcal{A}_n(\alpha, \beta)| = 1 + \binom{n}{2} + \binom{n+1}{5}$*

**PROPOSITION 3.4** *If $\alpha = 132$ and $\beta = 4321$ then $\mathcal{A}(\alpha, \beta)$ is the set of permutations whose profiles are in the profile closure of 32415 and 42135. Moreover, $|\mathcal{A}_n(\alpha, \beta)| = 1 + \binom{n+1}{3} + 2\binom{n}{4}$*

PROOF Note first that the profile closure of 32415 and 42135 is the set of profiles

$$P = \{32415, 42135, 3214, 3241, 4213, 213, 321, 21, 1\}$$

It is straightforward to verify that any permutation whose profile is in $P$ must avoid both 132 and 4321. To prove that any permutation of length $n$ which avoids both 132 and 4321 has profile in $P$ we argue by induction on $n$. Let $\sigma'$ be the permutation obtained by removing $n$ from $\sigma$. By induction, the profile of $\sigma'$ is one of the profiles in $P$. We shall consider the different possibilities for the profile of $\sigma'$ and verify that when $n$ is inserted into such a permutation to produce a permutation that avoids both 132 and 4321 then the result has a profile that is also in $P$.

1. $\sigma' = \gamma_3\gamma_2\gamma_4\gamma_1\gamma_5$. This is the case that $\sigma'$ has profile 32415; each $\gamma_i$ is an increasing sequence of consecutive symbols and the subscripts indicate the relative values of symbols in different $\gamma_i$. Notice that $n$ cannot be inserted in the interior of any $\gamma_i$ since that would introduce a subsequence order isomorphic to 132 (this observation applies to all the cases). Also $n$ cannot be inserted before $\gamma_3$ since that would introduce a subsequence order isomorphic to 4321. Nor can $n$ be inserted anywhere between $\gamma_3$ and $\gamma_5$ for that would produce a subsequence order isomorphic to 132. So the only valid place where $n$ can be inserted is after $\gamma_5$ and then the result also has profile 32415

2. $\sigma' = \gamma_4\gamma_2\gamma_1\gamma_3\gamma_5$. Again we need only consider insertion points for $n$ which fall between $\gamma$-strings and, just as above, the only possible place is at the end of $\sigma'$ giving a permutation also of profile 42135.

3. $\sigma' = \gamma_3\gamma_2\gamma_1\gamma_4$. The argument is exactly the same.

4. $\sigma' = \gamma_3\gamma_2\gamma_4\gamma_1$. To avoid introducing a subsequence order isomorphic to 4321 or 132 the only possible places to insert $n$ are between $\gamma_4$ and $\gamma_1$, or after $\gamma_1$. The former yields a permutation with profile 3214 and the latter yields a permutation with profile 32415.

5. $\sigma' = \gamma_4\gamma_2\gamma_1\gamma_3$. Here the valid insertion points are between $\gamma_4$ and $\gamma_2$ which gives the profile 4213, and after $\gamma_3$ giving the profile 42135.

For $\sigma'$ of the form $\gamma_2\gamma_1\gamma_3, \gamma_3\gamma_2\gamma_1, \gamma_2\gamma_1, \gamma_1$ the argument is similar.

Finally, we apply Lemma 2.3 to each of the profiles in $P$. This shows that $|\mathcal{A}_n(\alpha,\beta)| = 2\binom{n-1}{4} + 3\binom{n-1}{3} + 2\binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0} = 1 + \binom{n+1}{3} + 2\binom{n}{4}$ ■

**PROPOSITION 3.5** *If $\alpha = 123$ and $\beta = 4213$ then $\mathcal{A}_n(\alpha,\beta) = 3 \times 2^{n-1} - \binom{n+1}{2} - 1$*

Proof It is convenient to consider the equivalent problem of finding $|\mathcal{A}_n(321, 3124)|$. We begin by considering permutations which have a decomposition into segments $\phi_1\phi_2\ldots\phi_r$ where $\phi_i < \phi_{i+1}$, and each $\phi_i$ is order isomorphic to $[2, 3, \ldots, k, 1]$ with $k = |\phi_i|$. For convenience we call such permutations *$\phi$-permutations.* Note that we allow $|\phi_i| = 1$ in which case $\phi_i$ is called *trivial.* Actually, the set of $\phi$-permutations is precisely $\mathcal{A}(321, 312)$ although we shall not need this fact.

Given a permutation $\sigma \in \mathcal{A}(321, 3124)$ we write $\sigma = \gamma\delta$ where $\gamma$ is the longest initial segment of $\sigma$ which is a $\phi$-permutation. We shall now prove, by induction on $n$, that $\gamma < \delta$ and that $\sigma$ has one of the following forms:

(a) $\gamma$

(b) $\gamma\delta_2\delta_1$, with $|\delta_1| > 1$

(c) $\gamma\delta_2\delta_4\delta_1^{(1)}\delta_3$

(d) $\gamma\delta_3\delta_1^{(1)}\delta_4\delta_2$

(e) $\gamma\delta_2\delta_4\delta_1^{(1)}\delta_5\delta_3$

where each $\delta_i$ is a non-empty increasing sequence of consecutive integers, the superscript (1) signifies that the length of $\delta_i$ is 1, and $\delta_i < \delta_{i+1}$.

To carry out the inductive step we consider a permutation $\sigma = \gamma\delta$ of length $n-1$ where $\gamma < \delta$ and $\sigma$ has one of the 5 forms above. We shall prove that if $n$ is inserted anywhere within $\sigma$, without introducing a subsequence order isomorphic to 321 or 3124, then the resulting sequence is also of this type. We shall consider each case in turn.

(a) $\sigma = \gamma$, i.e. $\sigma = \phi_1\phi \dots \phi_r$ is a $\phi$-permutation. If $|\phi_i| \geq 2$ the final two symbols of $\phi_i$ are decreasing and, in order to avoid introducing a subsequence order isomorphic to 321, $n$ must not be inserted before the first of these two symbols. Therefore the only valid insertion points for $n$ are immediately before the last symbol of the final non-trivial $\phi_i$ or at a later place than this.

If $n$ were to be inserted between symbols $s, t$ of $\phi_i = [t+1, t+2, \dots, s, t]$ (where $\phi_{i+1}, \dots, \phi_r$ are all trivial) then we would obtain

$$[\phi_1\phi_2 \dots \phi_{i-1}, t+1, t+2, \dots, s, n, t, s+1, s+2, \dots, n-1]$$

and then with $[\phi_1\phi_2 \dots \phi_{i-1}]$, $[t+1, t+2, \dots, s]$,$[n]$,$[t]$,$[s+1, s+2, \dots, n-1]$ in the roles of $\gamma, \delta_2, \delta_4, \delta_1^{(1)}, \delta_3$ respectively we would have a permutation of type (c).

If $n$ were to be inserted in a later position we would obtain one of

$$[\phi_1\phi_2 \dots \phi_i, u, u+1, \dots, n-2, n-1, n]$$

$$[\phi_1\phi_2 \dots \phi_i, u, u+1, \dots, n-2, n, n-1]$$

$$[\phi_1\phi_2 \dots \phi_i, u, u+1, \dots, v, n, v+1, \dots, n-2, n-1] \text{ with } v+1 < n-1$$

In the first two cases we again have a permutation of type (a) while in the third case we have a permutation of type (b) (with $[\phi_1\phi_2 \dots \phi_i, u, u+1, \dots, v]$ in the role of $\gamma$, $[n]$ in the role of $\delta_2$, and $[v+1, \dots, n-2, n-1]$ in the role of $\delta_1$).

(b) $\sigma = \gamma\delta_2\delta_1$ with $\delta_1 = b_1b_2 \dots b_u$ and $u \geq 2$. In this case (and all further cases) $n$ is necessarily inserted within $\delta$ to avoid having a subsequence order isomorphic to 321. In fact, only two insertion points are valid. If $n$ is inserted before the final symbol of $\delta_2$ a subsequence order isomorphic to 321 would be created. On the other hand, if $n$ is inserted after the second symbol of $\delta_1 = b_1b_2 \dots$ we would obtain a subsequence $xb_1b_2n$ which is order isomorphic to 3124 (here $x \in \delta_2$). The two possible insertion points are therefore between $\delta_2$ and $\delta_1$ which leads to a permutation of type (b) again; or between $b_1$ and $b_2$ which leads to a permutation of type (d) (with $\delta_2, [b_1], [n], [b_2, \dots, b_u]$ in the roles of $\delta_3, \delta_1, \delta_4, \delta_2$ respectively).

(c) $\sigma = \gamma\delta_2\delta_4\delta_1^{(1)}\delta_3$. To avoid introducing a subsequence order isomorphic to 321 $n$ cannot be inserted before the final symbol of $\delta_4$. Also, to avoid introducing a subsequence order isomorphic to 3124 $n$ cannot be inserted after the first symbol of $\delta_3$. The only possible insertion points are therefore between $\delta_4$ and $\delta_1^{(1)}$ which gives a permutation of type (c) again; or between $\delta_1^{(1)}$ and $\delta_3$ which gives a permutation of type (e).

Cases (d) and (e) are handled in a similar way.

Next observe that the classification of $\mathcal{A}(321, 3124)$ into the 5 types above is a disjoint partition of $\mathcal{A}(321, 3124)$ (this is why the condition $|\delta_1| > 1$ is needed for type (b):— if $|\delta_1| = 1$ then $\delta_2\delta_1$ would have the form of one of the segments $\phi_i$ of $\gamma$ and so $\gamma$ would not be a longest initial $\phi$-permutation of $\sigma$).

Thus $|\mathcal{A}_n(321, 3124)|$ can be obtained by enumerating the permutations of length $n$ of each of the 5 types and summing the results. This is easily done.

For example (the most complicated case) permutations $\gamma\delta_2\delta_4\delta_1^{(1)}\delta_5\delta_3$ of length $n$ (type (e)) are completely determined by the lengths of $\phi_1, \ldots, \phi_r$ (where $\gamma = \phi_1 \ldots \phi_r$) and the lengths of $\delta_2, \delta_4, \delta_5, \delta_3$. These are a set of at least 4 positive integers whose sum is $n-1$, and every such set of integers gives rise to a permutation of length of type (e). Therefore there are $2^{n-2}-1-(n-2)-\binom{n-2}{2}$ such permutations. Carrying out a similar analysis in all the other cases leads to the required result. ■

**PROPOSITION 3.6** *1.* $\mathcal{A}(321, 2143) = \mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$

*2.* $\mathcal{A}(321, 2143, 3142) = S_2^{-1}$

*3.* $\mathcal{A}(321, 2143, 3142)^{-1} = \mathcal{A}(321, 2143, 2413) = S_2$

*4.* $|\mathcal{A}_n(321, 2143, 3142)| = 2^n - n$

*5.* $|\mathcal{A}_n(321, 2143)| = 2^{n+1} - 2n - 1 - \binom{n+1}{3}$

Proof For part 1 we can confirm, by case checking, that any permutation $\tau$ which involves 3142 and 2413 necessarily involves 321 or 2143; only a finite number of cases have to be checked since we may presume that $\tau$ is a minimal permutation involving 3142 and 2413 (and so of length at most 8). This implies that a permutation which avoids 321 and 2143 must avoid at least one of 3142 and 2413.

For part 2 it is easy to see that the right-hand side set is contained in the left-hand side set. Now let $\sigma \in \mathcal{A}(321, 2143, 3142)$ and write

$$\sigma = [1, 2, \ldots, m, a_1, a_2, \ldots, a_r, m+1, b_1, \ldots, b_s]$$

where $m \geq 0$ and $r \geq 1$. Since every $a_i > m+1$ and $\sigma$ avoids 321 $a_1 < a_2 < \ldots < a_r$. Moreover $b_1, \ldots, b_s$ must also be increasing since, if $b_i > b_{i+1}$ then the subsequence $[a_1, m+1, b_i, b_{i+1}]$ is either order isomorphic to 4132 which involves 321 if $a_1 > b_i$, or order isomorphic to 3142 if $b_i > a_1 > b_{i+1}$, or order isomorphic to 2143 if $b_{i+1} > a_i$. Thus, $\sigma = \gamma\delta$ where $\gamma, \delta$ are increasing and so $\sigma \in S_2^{-1}$.

Part 3 is true because the permutation inverse of 3142 is 2413.

For part 4 a permutation $\sigma = \gamma\delta$ (with $\gamma, \delta$ increasing) of $\mathcal{A}_n(321, 2143, 3142)$ is defined once the subset of values in $\gamma$ is determined. However, although there are $2^n$ such subsets, $n+1$ of them (those of the form $\{1, 2, \ldots, i\}$) all give the same permutation $\sigma$ and so there are $2^n - n$ such permutations.

Finally, to prove part 5 we use

$$\begin{array}{rl} |\mathcal{A}_n(321, 2143, 3142) \cup \mathcal{A}_n(321, 2143, 2413)| & = \\ |\mathcal{A}_n(321, 2143, 3142)| + |\mathcal{A}_n(321, 2143, 2413)| & - \\ |\mathcal{A}_n(321, 2143, 3142) \cap \mathcal{A}_n(321, 2143, 2413)| & \end{array}$$

However, by Theorem 2.7, $\mathcal{A}(321, 2143, 3142) \cap \mathcal{A}(321, 2143, 2413) = S_2 \cap S_2^{-1}$ and so, by Lemma 2.8, this means that

$$|\mathcal{A}_n(321, 2143, 3142) \cap \mathcal{A}_n(321, 2143, 2413)| = \binom{n+1}{3} + 1$$

The result now follows using part 4. ■

The next proposition sees the first application of a useful fact about permutations avoiding 132. Such a permutation may be written as $\gamma n \delta$, where $n$ is the largest symbol. Since, for every $c \in \gamma$ and $d \in \delta$, $cnd$ is not order isomorphic to 132 we have $c > d$; in other words $\gamma > \delta$.

**PROPOSITION 3.7** *If* $\alpha = 132$ *and* $\beta = 4312$ *then* $|\mathcal{A}_n(\alpha, \beta)| = (n-1)2^{n-2} + 1$

Proof Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ with $i = |\gamma|, j = |\delta|$ and $i + j = n - 1$. Since $132 \not\preceq \sigma$, we have $\gamma > \delta$. Also $\delta$ avoids 312 (since $4312 \preceq \sigma$). We consider two cases. Suppose first that $\delta$ is not decreasing (and so $j \geq 2$). Then, since $4312 \not\preceq \sigma$, $\gamma$ must be increasing and so all of the $2^{j-1} - 1$ non-decreasing permutations of $\mathcal{A}_j(132, 312)$ (see [10], Proposition 10) are possibilities for $\delta$. This gives

$$\sum_{j=2}^{n-1} \left(2^{j-1} - 1\right) = 2^{n-1} - n$$

possibilities for $\sigma$. On the other hand, if $\delta$ is decreasing, every one of the permutations of $j+1, \ldots, j+i$ which avoid both 132 and 4312, of which there $a_i$, can arise as a possibility for $\gamma$. This gives a further $\sum_{i=0}^{n-1} a_i$ possibilities for $\sigma$. Hence

$$a_n = \sum_{i=0}^{n-1} a_i + 2^{n-1} - n$$

By differencing we find $a_n = 2a_{n-1} + 2^{n-2} - 1$ from which the result follows by induction. ■

**PROPOSITION 3.8** *If* $\alpha = 132$ *and* $\beta = 4231$ *then* $|\mathcal{A}_n(\alpha, \beta)| = (n-1)2^{n-2} + 1$

Proof Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ with $i = |\gamma|, j = |\delta|$ and $i + j = n - 1$. Since $132 \not\preceq \sigma$, we have $\gamma > \delta$. Also $\delta$ avoids 231 (since $4231 \preceq \sigma$). We consider two cases. If $j = 0$ then $\gamma$ can be any permutation of $\mathcal{A}_{n-1}(\alpha, \beta)$ of which there are $a_{n-1}$. If $j \neq 0$ then, as $4231 \not\preceq \sigma$, $\gamma$ must avoid 312. Conversely, if $\gamma$ avoids 132 and 312, and $\delta$ avoids 312 and 231, then $\sigma$ will avoid $\alpha$ and $\beta$. By [10], Propositions 10 and 8, there are $2^{i-1}$ choices for $\gamma$ if $i \geq 1$ (1 if $i = 0$) and

$2^{j-1}$ choices for $\delta$. For each $i = 1, \dots, n-2$ we therefore have $2^{i-1}2^{j-1} = 2^{n-3}$ possibilities for $\sigma$ ($2^{n-2}$ if $i = 0$). Hence

$$a_n = a_{n-1} + 2^{n-2} + \sum_{i=1}^{n-2} 2^{n-3} = a_{n-1} + n2^{n-3}$$

from which the result follows by induction. ■

**PROPOSITION 3.9** *If $\alpha = 132$ and $\beta = 3214$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 4a_{n-1} - 5a_{n-2} + 3a_{n-3}$.*

PROOF Let $\sigma = \gamma n \delta \in \mathcal{A}_n(132, 3214)$, and let $j = |\delta|$ with $0 \leq j < n$. Since $132 \not\preceq \sigma$, $\gamma > \delta$. Thus $\delta$ is a permutation on $1, 2, \dots, j$ and so $\delta \in \mathcal{A}_j(132, 3214)$. Also, because $3214 \not\preceq \sigma$, $\gamma$ is a sequence on $j+1, \dots, n-1$ order isomorphic to a permutation which involves neither 132 nor 321 and, by [10] Proposition 11, there are $\binom{n-1-j}{2} + 1$ such permutations. Hence

$$a_n = \sum_{j=0}^{n-1} \left( \binom{n-1-j}{2} + 1 \right) a_j$$

Differencing this recurrence three times gives the required result. ■

**PROPOSITION 3.10** *If $\alpha = 123$ and $\beta = 3214$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

PROOF If $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ then $\gamma$ will be decreasing (to avoid 123) and $|\gamma| \leq 2$ (to avoid 3214). Let $x_n, y_n, z_n$ be, respectively, the number of possibilities for $\sigma$ of the form $n\delta, an\delta, abn\delta$ (with $a > b$). Thus $|\mathcal{A}_n(\alpha, \beta)| = x_n + y_n + z_n$. If $\sigma = n\delta$, $\delta$ can be any permutation in $\mathcal{A}_{n-1}(\alpha, \beta)$ and so

$$x_n = x_{n-1} + y_{n-1} + z_{n-1}$$

If $\sigma = an\delta$, again $a\delta$ can be any permutation in $\mathcal{A}_{n-1}(\alpha, \beta)$ and so

$$y_n = x_{n-1} + y_{n-1} + z_{n-1}$$

If $\sigma = abn\delta$ then $ab\delta \in \mathcal{A}_{n-1}(\alpha, \beta)$ but $b \neq n-1$ and so

$$z_n = x_{n-1} + z_{n-1}$$

Solving these recurrences and using $a_n = x_n + y_n + z_n$ gives the result. ■

**PROPOSITION 3.11** *If $\alpha = 132$ and $\beta = 1234$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof It is convenient to consider, instead, the equivalent pair $\alpha = 213$, $\beta = 1234$. If $\sigma = \gamma n \delta$ then $\gamma$ will be increasing (to avoid 213) with $|\gamma| \leq 2$ (to avoid 1234). Let $x_n, y_n, z_n$ be defined as in the previous Proposition. Then

$$x_n = x_{n-1} + y_{n-1} + z_{n-1}$$

$$y_n = x_{n-1} + y_{n-1} + z_{n-1}$$

$$z_n = y_{n-1} + z_{n-1}$$

(the last equation because $a = n-1$ is impossible). Again, solving these recurrences gives the result. ■

**PROPOSITION 3.12** *If $\alpha = 132$ and $\beta = 4213$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ and let $j = |\delta|$. Since $132 \not\preceq \sigma$ we have $\gamma > \delta$. Thus $\delta$ is a permutation of $1, 2, \ldots, j$ which avoids 132 and 213 and, by [10] Proposition 8, there are $2^{j-1}$ such permutations if $j > 0$ (and 1 if $j = 0$). Since there are $a_{n-j-1}$ choices for $\gamma$ we have

$$a_n = \sum_{j=1}^{n-1} 2^{j-1} a_{n-j-1} + a_{n-1}$$

Hence

$$\begin{aligned} a_n - 2a_{n-1} &= \sum_{j=1}^{n-1} 2^{j-1} a_{n-j-1} + a_{n-1} - 2\sum_{j=1}^{n-2} 2^{j-1} a_{n-j-2} - 2a_{n-2} \\ &= a_{n-1} - a_{n-2} \end{aligned}$$

as required. ■

**PROPOSITION 3.13** *If $\alpha = 132$ and $\beta = 4123$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof The argument follows the same lines as the previous proof. Here $\delta$ is a permutation of $1, 2, \ldots, j$ which avoids 132 and 213 and, by [10] Proposition 7, there are $2^{j-1}$ such permutations if $j > 0$ (1 if $j = 0$). ■

**PROPOSITION 3.14** *If $\alpha = 132$ and $\beta = 3124$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof As in the last two propositions we let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ where $\gamma > \delta$. This time however we note that $\gamma$ avoids 132 and 312 and so, by [10] Proposition 10, there are $2^{i-1}$ choices for $\gamma$ if $|\gamma| = i$. So again

$$a_n = \sum_{i=1}^{n-1} 2^{i-1} a_{n-i-1} + a_{n-1}$$

■

**PROPOSITION 3.15** *If $\alpha = 123$ and $\beta = 2143$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ with $\gamma = [g_1, g_2, \ldots, g_i]$. If $i = 0$ or $i = 1$ then $\sigma \in \mathcal{A}_n(\alpha, \beta)$ if and only if $\gamma\delta \in \mathcal{A}_{n-1}(\alpha, \beta)$ so these cases each contribute $a_{n-1}$ to $|\mathcal{A}_n(\alpha, \beta)|$. If $i \geq 2$ then $[g_1, g_2, \ldots, g_i]$ is decreasing (otherwise $123 \preceq \sigma$) and $g_{i-1} > d$ for all $d \in \delta$ (otherwise $[g_{i-1}, g_i, n, d]$ will be order isomorphic to 2143). Thus

$$\sigma = [n-1, n-2, \ldots, n-i+1, g_i, n, \delta]$$

Such permutations are in one-to-one correspondence with permutations $[g_i, \delta]$ in $\mathcal{A}_{n-i}(\alpha, \beta)$ and so there are $a_{n-i}$ for each $i \geq 2$. Consequently

$$a_n = 2a_{n-1} + \sum_{i=2}^{n-1} a_{n-i}$$

and the result follows by differencing. ■

**PROPOSITION 3.16** *If $\alpha = 123$ and $\beta = 3142$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

Proof Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ with $\gamma = [g_1, \ldots, g_i]$. Then $\gamma$ is decreasing (since $\sigma$ avoids 123) and $g_1, \ldots, g_i$ are consecutive integers (if $g_i > u > g_{i+1}$ then $u \in \delta$ and $[g_i, g_{i+1}, n, u]$ is order isomorphic to 3142).

If $i = 0$ then $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ if and only if $\delta \in \mathcal{A}_{n-1}(\alpha, \beta)$ so this case accounts for $a_{n-1}$ possibilities for $\sigma$. For each $i \geq 1$, $\sigma = [g_1, \ldots, g_i, n, \delta] \in \mathcal{A}_n(\alpha, \beta)$ if and only if $g_1\delta$ avoids $\alpha, \beta$. The sequence $g_1\delta$ has length $n - i$, its elements comprise the set $\{1, 2, \ldots, n\} \setminus \{g_1 - 1, \ldots, g_1 - i + 1, n\}$, and it is order isomorphic to a permutation $\theta$ on $\{1, 2, \ldots, n-i\}$. Moreover, for given $n, i$, every permutation $\theta$ on $\{1, 2, \ldots, n-i\}$ determines a unique order isomorphic sequence $g_1\delta$ on a set $\{1, 2, \ldots, n\} \setminus \{g_1 - 1, \ldots, g_1 - i + 1, n\}$. Hence the number of possibilities for $g_1\delta$ is $|\mathcal{A}_{n-i}(\alpha, \beta)|$. Therefore

$$a_n = a_{n-1} + \sum_{i=1}^{n-1} a_{n-i}$$

as in the previous proposition. ■

**PROPOSITION 3.17** *If $\alpha = 132$ and $\beta = 2134$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

PROOF It is convenient to consider, instead, the equivalent pair $\alpha = 312, \beta = 3421$. Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$ with $\delta = [d_1 d_2 \ldots, d_j]$. Then $\delta$ is decreasing (to avoid 312) and, for all $g \in \gamma$, $g < d_{j-1}$ (otherwise $[g, n, d_{j-1}, d_j]$ is order isomorphic to 3421). Thus $\sigma$ has either the form $\gamma n$ or the form $[\gamma, n, n-1, \ldots n-(j-1), d_j]$. In the first case, $\gamma n \in \mathcal{A}_n(\alpha, \beta)$ if and only if $\gamma \in \mathcal{A}_{n-1}(\alpha, \beta)$ and so there are $a_{n-1}$ permutations of this type. Permutations of the second type are in one-to-one correspondence with permutations $\gamma d_j \in \mathcal{A}_{n-j}(\alpha, \beta)$ and there are $a_{n-j}$ of these for each value of $j$. Hence

$$a_n = a_{n-1} + \sum_{j=1}^{n-1} a_{n-j}$$

as before. ■

**PROPOSITION 3.18** *If $\alpha = 132$ and $\beta = 3412$ the values of $a_n = |\mathcal{A}_n(\alpha, \beta)|$ satisfy the recurrence $a_n = 3a_{n-1} - a_{n-2}$.*

PROOF Let $\sigma = \gamma n \delta \in \mathcal{A}_n(\alpha, \beta)$. Then $\gamma > \delta$ as $132 \not\preceq \sigma$. If $\gamma$ is empty then $\delta$ can be any of the $a_{n-1}$ members of $\mathcal{A}_{n-1}(\alpha, \beta)$. If $\gamma$ is non-empty then, in order to avoid $3412 \preceq \sigma$, $\delta$ must be decreasing. In this case $\gamma$ avoids both 132 and 3412 and for each $i = |\gamma|$ there are $a_i$ such permutations. This gives

$$a_n = a_{n-1} + \sum_{i=1}^{n-1} a_i$$

again. ■

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